15/03/2007

BIOL/PHYS 438

Zoological Physics

- **Logistics** (Next week: more *ACOUSTICS*)
- *Review* of **ELECTRO**MAGNETISM
  - Phenomenology of Q & $\boldsymbol{E}$: **Coulomb/Gauss**
  - Electrostatic Potential V ("Voltage")
  - Batteries & Capacitors : cell membranes
  - Conductors & Resistance R
  - RC circuits & time constants

    (Also next week: ELECTRO**MAGNET**ISM)

Logistics

Assignment 1: Solutions now online!

Assignment 2: Solutions now online!

Assignment 3: Solutions now online!

Assignment 4: Solutions online soon!

Assignment 5: due **Today**

Assignment 6: due **Thursday after next**

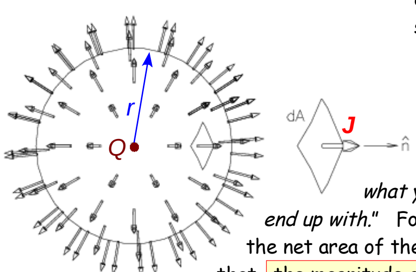
Hopefully your **Projects** are *well* underway now . . .

Conservation & Symmetry

Think of Q as the source of a flux $\boldsymbol{J}$ of some indestructible "stuff" (water, energy, anything that is *conserved*) so that $\boldsymbol{J}$ points away from Q in all directions ("isotropically").

By **symmetry**, $\boldsymbol{J}$ must be normal to the surface of a sphere centred on Q and have the same magnitude J everywhere on the sphere's surface. **Gauss' Law** says:

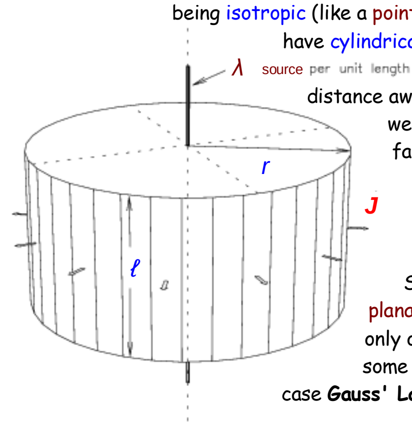
$$\oint\!\!\oint_S \boldsymbol{J} \cdot d\vec{A} = Q_{\rm encl}$$



. . . *i.e.* "*In steady state, what you start with is what you end up with.*" For an isotropic source, since the net area of the sphere is $4\pi r^2$, this says that the magnitude of $\boldsymbol{J}$ falls off as $1/r^2$.

Cylindrical Symmetry

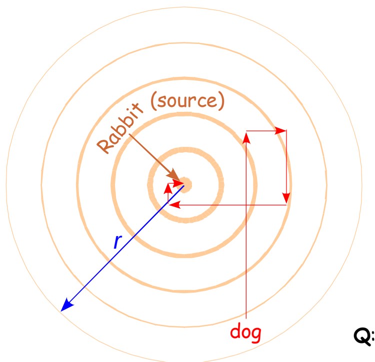
There is nothing in the preceding arguments that depends upon the source being isotropic (like a point source). It could equally well have cylindrical symmetry (like a line source), depending only on r, the distance away from the line. In this case we get a flux $\boldsymbol{J}$ whose magnitude falls off like $1/r$ instead of $1/r^2$.



$$J \propto 1/r$$

Similar arguments apply for a planar source (one which depends only on the distance r away from some plane of symmetry). In that case **Gauss' Law** predicts *no falloff at all*!

Predators know Gauss' Law!



"lines of rabbit"

Strategy: head in a straight line as long as the local magnitude of "rabbit flux" is *increasing*. When it starts to *decrease*, make a 90° *right turn* [or left, but *always the same way!*]. **Repeat**.

This will always lead you to the rabbit, unless it realizes your strategy and moves.

Q: is there any *better* strategy?

Q: what would a *really clever* rabbit do?

Coulomb's Law

$$\vec{F}^E_{12} = k_E \frac{q_1 q_2}{r_{12}^2} \hat{r}_{12}$$

Think of q_1 as the source of "electric field lines" $\boldsymbol{E}$ pointing away from it in all directions. (For a + charge. A – charge is a sink.)

Then $\boldsymbol{F}_{12} = q_2 \boldsymbol{E}$ where we think of $\boldsymbol{E}$ as a vector field that is "just there for some reason" and q_2 is a "test charge" placed at some position where the effect ($\boldsymbol{F}$) of $\boldsymbol{E}$ is manifested.
We can then write Coulomb's Law a bit more simply:

$$\vec{E} = k_E \frac{Q}{r^2} \hat{r}$$

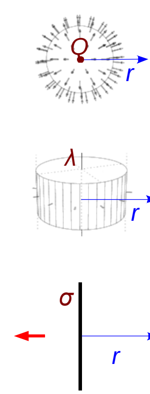
Fundamental Constants

$c \equiv 2.99792458 \times 10^8$ m/s *exactly!*

$k_E \equiv 1/4\pi\epsilon_0 = c^2 \times 10^{-7} \approx 8.98755 \times 10^9$ V·m·C^{-1}
$= 8.987551787368176 \times 10^9$ V·m·C^{-1} *exactly!*

$\epsilon_0 = 10^7 / 4\pi c^2 \approx 8.8542 \times 10^{-12}$ C^2·N^{-1}·m^{-2}

Electric Fields



$$\vec{E}_{\rm sph} = \frac{Q}{4\pi\epsilon} \frac{\hat{r}}{r^2}$$

$$\vec{E}_{\rm cyl} = \frac{\lambda}{2\pi\epsilon} \frac{\hat{r}}{r}$$

$$\vec{E}_{\rm plane} = \frac{\sigma}{2\epsilon} \hat{r}$$

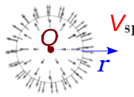
(independent of r)

$\epsilon = k\,\epsilon_\circ$,
where k is the dielectric constant.
In free space,

$\epsilon = \epsilon_\circ$.

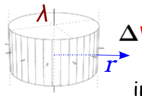
This automatically takes care of the effect of **dielectrics**.

Electrostatic Potentials



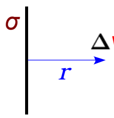
$$V_{\rm sph} = \frac{Q}{4\pi\epsilon}\frac{1}{r}$$

relative to $V \to 0$ as $r \to \infty$



$$\Delta V_{\rm cyl} = \frac{\lambda}{2\pi\epsilon}\log\left(\frac{r_0}{r}\right)$$

in moving from r_0 to r



$$\Delta V_{\rm plane} = \frac{\sigma}{2\epsilon}(r_0 - r)$$

in moving from r_0 to r

$dV = -\mathbf{E} \cdot d\mathbf{r}$

$\mathbf{E} = -\vec{\nabla} V$

where

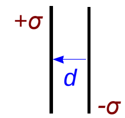
$$\vec{\nabla} \equiv \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

For finite objects,

$\lambda = Q/L$

$\sigma = Q/A$

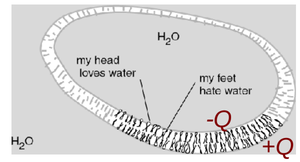
Model Cell Membrane



Two oppositely charged parallel plates of area A a distance d apart have a potential difference $\Delta V = Q\,d/\epsilon A$ when they carry a net charge of $\pm Q$ per plate.

Thickness of cell membrane: $d \approx 7$ nm

Surface area of a typical cell: $A \approx 3\text{x}10^{-10}$ m^2

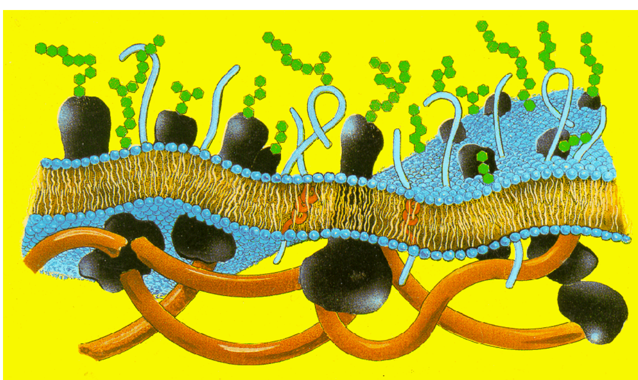


Dielectric constant: $k \approx 8.8$

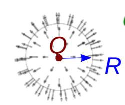
$\Delta V = -0.07$ V due to a negative charge of $Q \approx -0.245\text{x}10^{-12}$ Cb inside,

giving an electric field of $E = -\Delta V/d \approx 10^7$ V/m across the lipid bilayer.

"Actual" Cell Membrane

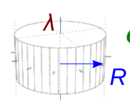


Capacitances



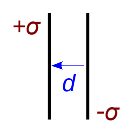
$$C_{\rm sph} = 4\pi\epsilon\left[\frac{1}{R_0} - \frac{1}{R}\right]^{-1}$$

relative to a concentric sphere at $R_0 > R$



$$C_{\rm cyl} = \frac{2\pi\epsilon L}{\log\left(R_0/R\right)}$$

relative to a coaxial cylinder at $R_0 > R$



$$C_{\rm \|plates} = \frac{\epsilon A}{d}$$

between two oppositely charged parallel plates

Definition of **capacitance**:

$Q = C\,V$

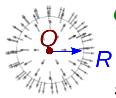
$V = Q/C$

$C = Q/V$

Note: each has the form

$C = (\epsilon)(\text{length})(\text{const.})$

Cell Membrane Capacitor

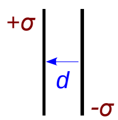


$$C_{\rm sph} = 4\pi\epsilon\left[\frac{1}{R_0} - \frac{1}{R}\right]^{-1}$$

relative to a concentric sphere at $R_0 > R$. *If* $R = R_0 + d$ *and* $d \ll R_0$, *then (approximately)*

$$C_{\rm sph} = 4\pi\epsilon\, R_0^2/d$$

which, with $A = 4\pi R_0^2$, *is the same as*



$$C_{\|\rm plates} = \frac{\epsilon A}{d}$$

between two oppositely charged parallel plates a distance d apart.

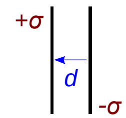
Thickness of cell membrane: $d \approx 7$ nm

Radius of a typical cell: $R_0 \approx 5\ \mu$m

$k \approx 8.8$

$C \approx 3.5\text{x}10^{-12}$ farads

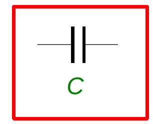
Capacitors



$$C_{\|\rm plates} = \frac{\epsilon A}{d}$$

between two oppositely charged parallel plates

Since all capacitors behave the same, we might as well pretend they are all made from two flat parallel plates, since that geometry is so easy to visualize. Thus the conventional symbol for a capacitor in a **circuit** is just the side view of such a device:



Definition of **capacitance**:

$Q = C \cdot \Delta V$

$\Delta V = Q/C$

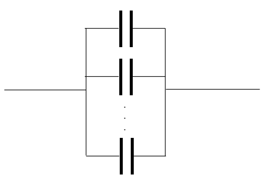
$C = Q/\Delta V$

where we now use the more conventional "ΔV" (for "voltage difference") instead of "Φ"

"Adding" Capacitors

<u>In **PARALLEL**</u>:

Same $\Delta V = Q_i/C_i$ across each C_i;
$Q_{\rm tot} = \Sigma_i Q_i = \Delta V\, \Sigma_i C_i$
or $C_{\rm eff} = \Sigma_i C_i$ -- *i.e.*
ADD CAPACITANCES!



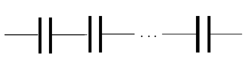
Definition of **capacitance**:

$Q = C \cdot \Delta V$

$\Delta V = Q/C$

$C = Q/\Delta V$

<u>In **SERIES**</u>:



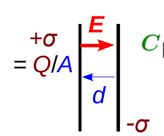
Charge is conserved $\Rightarrow$ same $\pm Q$ on each plate.
But $\Delta V = Q/C$ $\Rightarrow$ different ΔV_i across each C_i.
"Voltage drops" add up, giving $\Delta V_{\rm tot} = \Sigma_i Q/C_i$ or
$C_{\rm eff} = Q/\Delta V_{\rm tot} = 1/\Sigma_i C_i^{-1}$ -- *i.e.* **ADD INVERSES!**

Electrostatic Energy Storage

It takes electrical work $dW = V\,dQ$ to "push" a bit of charge dQ onto a capacitor C against the opposing EMF $V = -(1/C)\,Q$ (where Q is the charge already on the capacitor). This work is "stored" in the capacitor as $dU_E = -\,dW = (1/C)\,Q\,dQ$. If we start with an uncharged capacitor and add up the energy stored at each addition of dQ [*i.e.* integrate], we get

$$U_E = ½\,(1/C)\,Q^2$$

just like with a stretched spring -- $(1/C)$ is like a "spring constant".



$$C_{\|\rm plates} = \frac{\epsilon A}{d}$$

$E = \sigma/\epsilon = Q/A\epsilon$

or $Q = \epsilon A E$

Thus

$U_E = ½\,(d/\epsilon A)(\epsilon A E)^2$

$= ½\,(Ad)\,\epsilon E^2$

or

$$U_E/\text{Vol} \equiv u_E = ½\,\epsilon\, E^2$$

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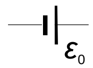
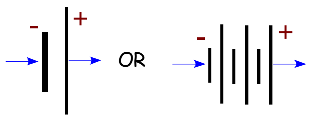
The Battery:



$\Delta V = + \mathcal{E}_0$

"**Voltage rise**" across a **battery**



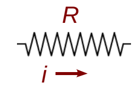
If we visualize **charge** as an incompressible **fluid** (like water) then the **battery** is like a **reservoir** stored at higher altitude than the circuit, providing a sort of "pressure head" to drive the fluid through the circuit. Such a flow of charge is called a "**current**", which nicely reinforces this metaphor.

Think of the **battery** as a constant (electromotive) "**force**" (EMF $\mathcal{E}_0$) that can be applied to a circuit.

This is pretty simple. Understanding how one **works** can be a bit more challenging.

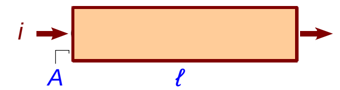
If we push the water into the rubber balloon (capacitor) it gets pushed back until the battery EMF is exactly balanced by the voltage drop across the capacitor. But there are other difficulties in pushing water through a pipe....

The Resistor:



$\Delta V = - i R$

"**Voltage drop**" across a resistor



The "incompressible fluid" flowing through a "**pipe**" experiences a "**drag force**" that is proportional to the **length** of the pipe and the **rate of flow** of the fluid, and **inversely** proportional to the cross-sectional **area** of the pipe. Analogously, the voltage drop across a resistor is proportional to its **length** and the current i and inversely proportional to its cross-sectional **area**. The constant of proportionality is called the **resistivity**, ρ :

$$R = \rho \, \ell / A$$

Think of the **resistor** as a **conduit** through which charge Q **flows** at a rate

$$i \equiv dQ/dt$$

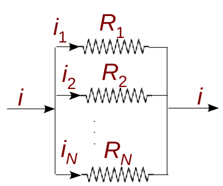
against an electromotive "**force**" caused by "**drag**".

The power P (rate of energy dissipation) in a resistor is given by

$$P = i \, \Delta V = i^2 R$$

"Adding" Resistors

In **PARALLEL**:



Same $\Delta V = - i_i R_i$ across each R_i ;

$i = - \Delta V / R_{\text{eff}}$

$= \Sigma_i i_i = - \Delta V \, \Sigma_i R_i^{-1}$

or $R_{\text{eff}} = (\Sigma_i R_i^{-1})^{-1}$ -- *i.e.* **ADD INVERSES!**

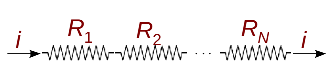
$\Delta V = - i R$

"**Voltage drop**" across a resistor

Kirchhoff's Laws:

- Charge Conserved: currents balance at any junction.
- V is single valued: voltage drops around any closed loop sum to zero.

In **SERIES**:



Charge is conserved $\Rightarrow$ same i through every R_i .
But different $\Delta V_i = - i R_i$ across each R_i.
"Voltage drops" add up, giving $\Delta V_{\text{tot}} = - i \, \Sigma_i R_i$ or
$R_{\text{eff}} = \Delta V_{\text{tot}} / i = \Sigma_i R_i$ -- *i.e.* just **ADD RESISTANCES!**

Properties of Air & Water

	Air	Pure Water	Sea Water	Fat
Dielectric Const. κ	1.00059	80.4	78 @ 0°C 70 @ 25°C	8.4
Resistivity ρ [Ω·m]	10^8	2×10^8	0.19	2.5×10^8

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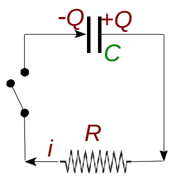
"Resistance is Futile!"

Discharging a **capacitor** through a **resistor**:



Kirchhoff's Law:
Sum of voltage drops around a circuit is zero.

We have $\Delta V_C = -Q/C$ for the charged capacitor.

Close switch at $t = 0$ with Q_0 on C. What happens?

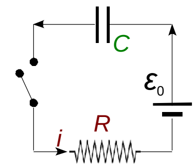
$i = dQ/dt$ begins to flow through R, causing a voltage drop $\Delta V_R = -iR$ across R.

Thus $-Q/C - iR = 0$, giving the differential equation $dQ/dt = -Q/RC$, which you should recognize instantly(!) as describing **exponential decay** (the rate of change of Q is negative and proportional to how much is left).

The answer (by inspection) is $Q(t) = Q_0 \exp(-t/\tau)$ where $\tau \equiv RC$ is the **time constant** for the decay.

"Charging a Capacitor"

Charging a **capacitor** through a **resistor**:



Kirchhoff's Law:
Sum of voltage drops around a circuit is zero.

We have an initially **un**charged capacitor. Close the switch at $t = 0$. What happens?

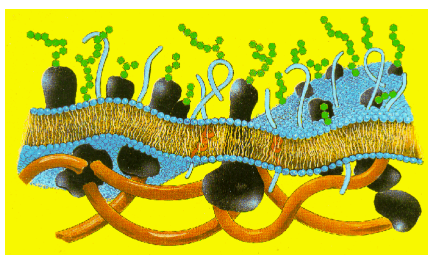
$i = dQ/dt$ begins to flow through R, causing a voltage drop $\Delta V_R = -iR$ across R.

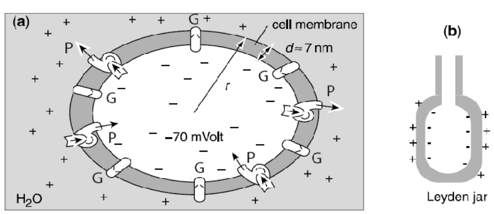
$\Delta V_C = -Q/C$ builds up on C.

Thus $\mathcal{E}_0 - Q/C - iR = 0$, giving $dQ/dt = \mathcal{E}_0/R - Q/RC$, which requires a change of variables to solve neatly: Let $x = \mathcal{E}_0/R - Q/\tau$ where $\tau \equiv RC$ as before. Then $dx/dt = -(1/\tau)\, dQ/dt = -(1/\tau)\, x$ so $x(t) = x_0\, e^{-t/\tau}$ where $x_0 = \mathcal{E}_0/R$. *Thus* $\mathcal{E}_0/R - Q(t)/\tau = (\mathcal{E}_0/R)\, e^{-t/\tau}$ *giving*

$Q(t) = C\mathcal{E}_0[1 - \exp(-t/\tau)]$ *since* $\tau\mathcal{E}_0/R = RC\,\mathcal{E}_0/R = C\mathcal{E}_0$.

Ion Pumps: Cells as Batteries





Na⁺ & K⁺ Pumps & Gates

(Recall CAP Lecture)