Simple Harmonic Motion

Many types of time-dependence

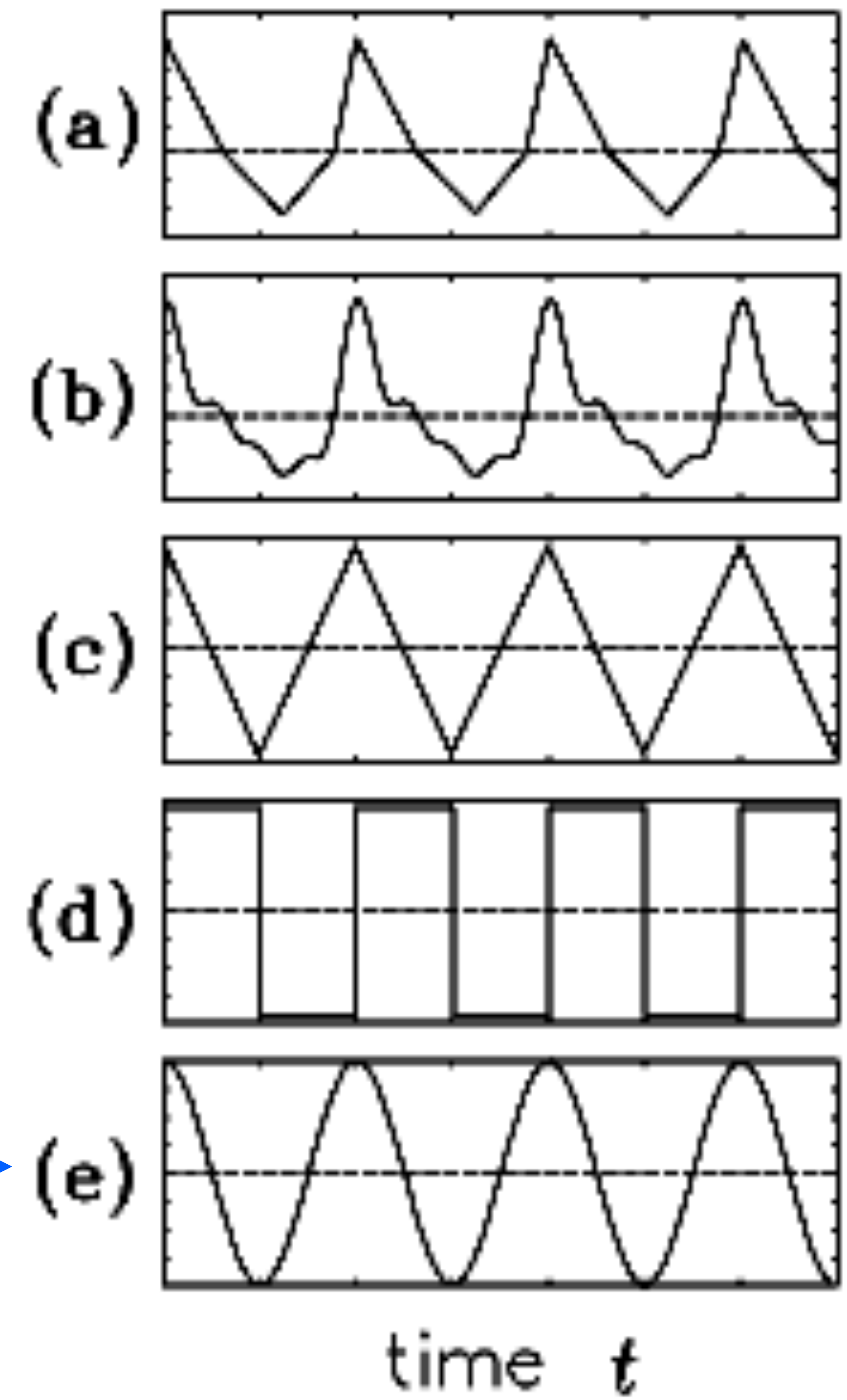
are ***periodic***,

but only one type is "***harmonic***",

namely,

***sinusoidal* motion.** →

What are its properties?

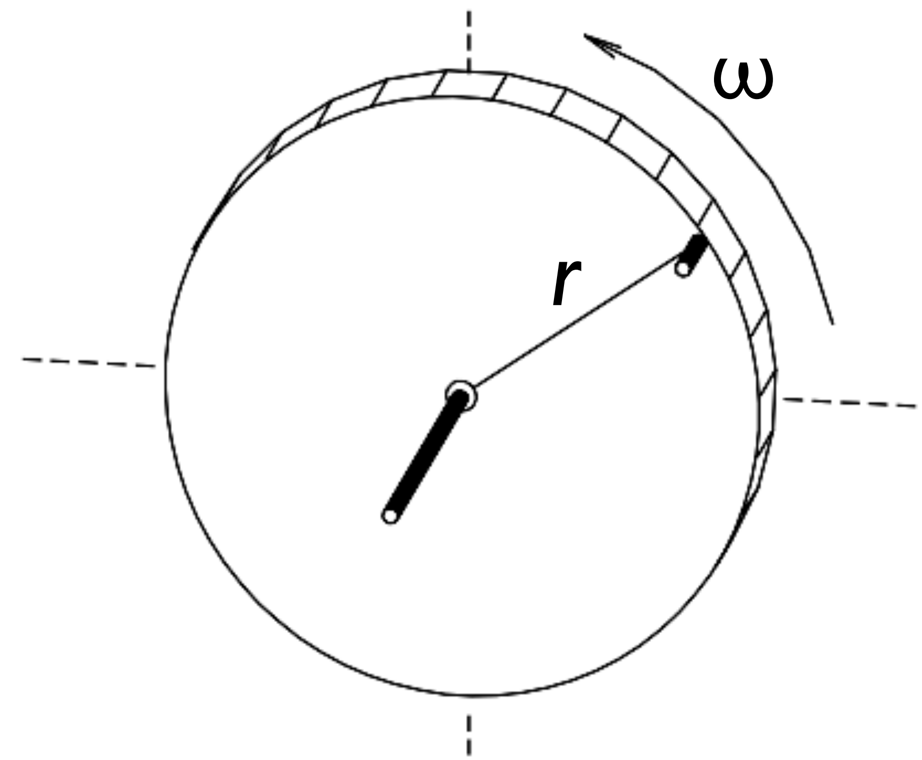


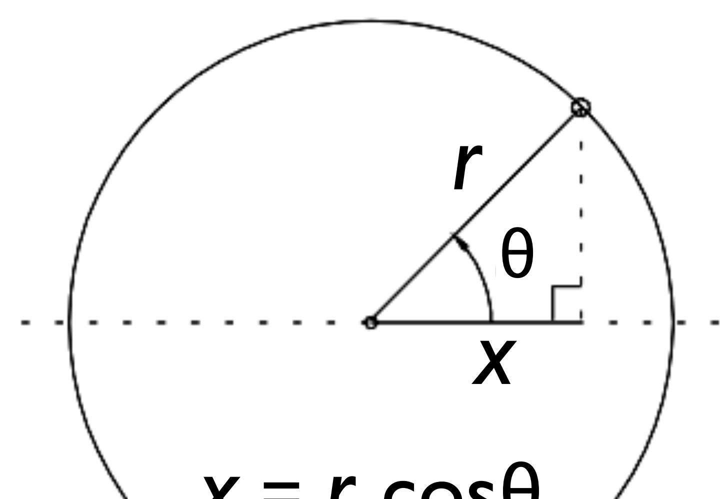
Projecting the Wheel

Picture a wheel spinning at constant angular velocity ω.

Now picture the motion of the ***shadow*** of a ***pin*** on the *rim* of the wheel (at high noon on the Equator).

This is called (reasonably) the ***projected*** *motion* of the pin.





$$x = r \cos\theta$$
$$\theta = \omega t$$

A Little Mathematics:

Small Angles:

$$\text{For} \quad \theta \ll 1, \quad \cos(\theta) \approx 1 - \tfrac{1}{2}\theta^2$$
$$\text{and} \quad \sin(\theta) \approx \theta.$$

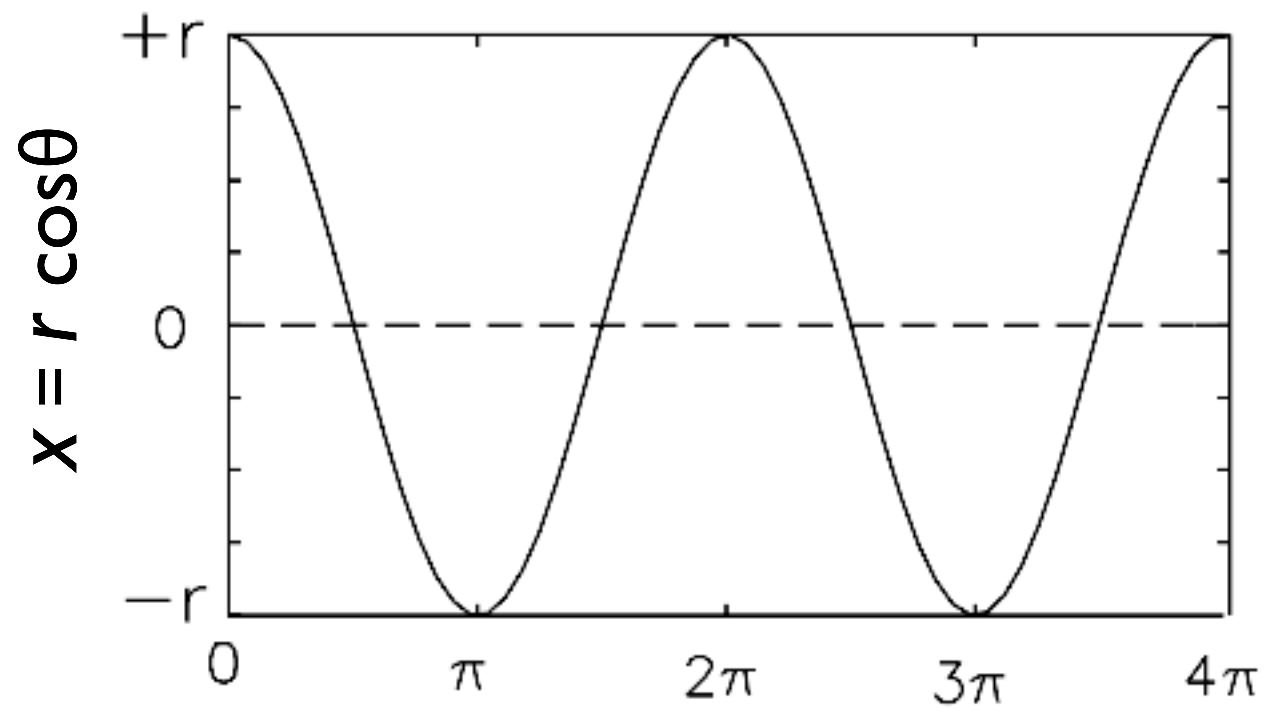
Taylor Series Expansions for Exponential & Sinusoidal Functions:

$$\exp(z) = 1 + z + \tfrac{1}{2}z^2 + \tfrac{1}{3!}z^3 + \tfrac{1}{4!}z^4 + \cdots$$

$$\cos(z) = 1 - \tfrac{1}{2}z^2 + \tfrac{1}{4!}z^4 - \cdots$$

$$\sin(z) = z - \tfrac{1}{3!}z^3 + \cdots$$

Derivatives of the Cosine Function



$$\theta = \omega t + \varphi$$

$$x = r\cos(\omega t + \varphi)$$

Note:

$$v \equiv dx/dt = -\omega r \sin(\omega t + \varphi)$$

$$d^2x/dt^2 = -\omega^2 x$$

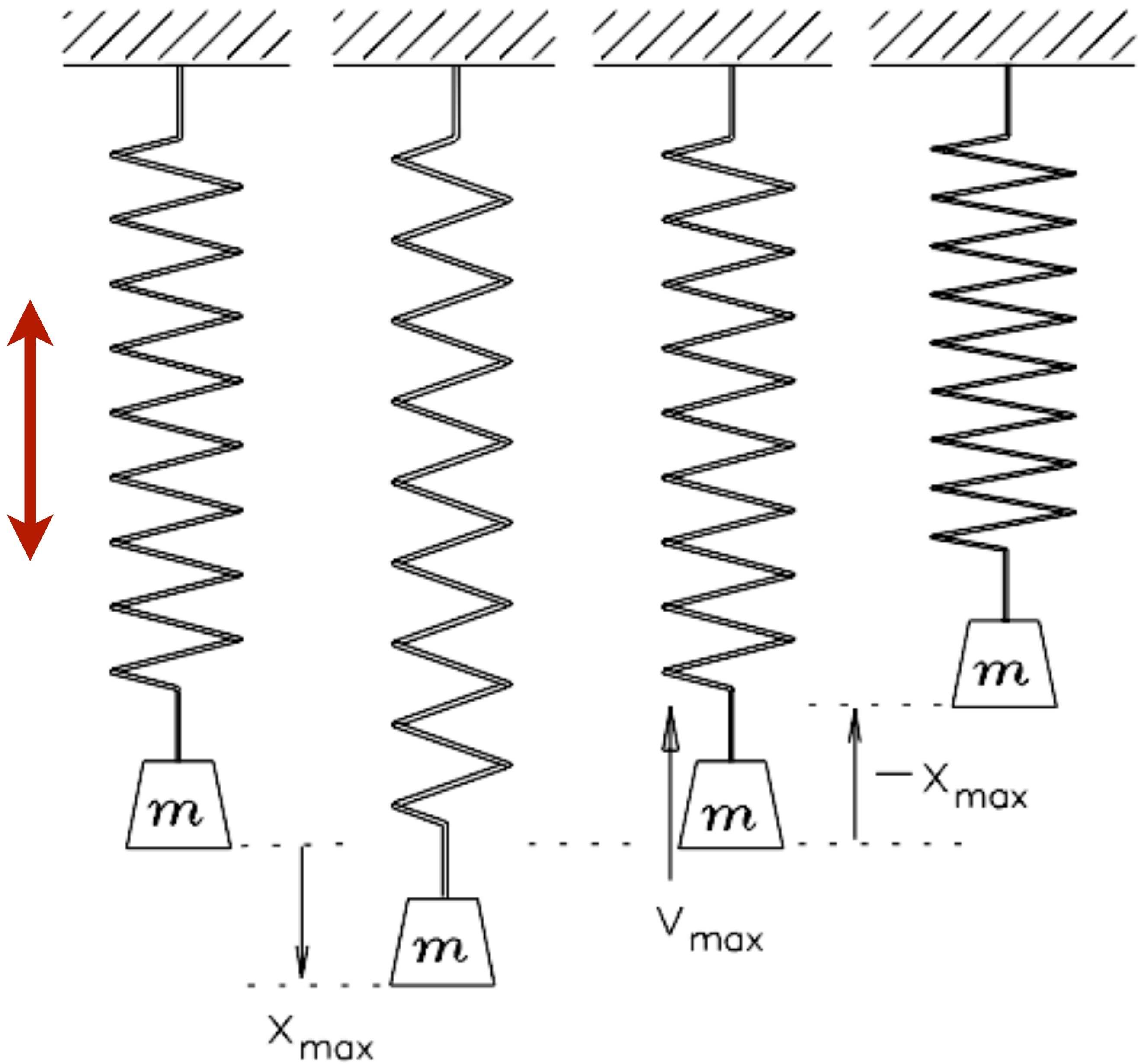
$$a \equiv d^2x/dt^2 = -\omega^2 r \cos(\omega t + \varphi)$$

The Spring Pendulum

$F = ma$

Hooke's Law $F = -kx$

$a = -(k/m)x$



So $a \equiv d^2x/dt^2 = -\omega^2 x$ if $\omega^2 = k/m$ or $\omega = \sqrt{k/m}$

Our old friend, the Exponential:

Remember $dx/dt = -\kappa x \Leftrightarrow x(t) = x_0 \exp(-\kappa t)$?

The second derivative would be $d^2x/dt^2 = \kappa^2 x$, right?

Well, now we have a new equation $d^2x/dt^2 = -\omega^2 x$, which means the *exponential* function would be a solution if only we could have $\kappa^2 = -\omega^2$.

Of course this is impossible. No real number is negative when squared.

But what if there *were* such a number? *Use your imagination!* $i = \sqrt{-1}$

Then we could solve our differential equation in one step:

$$x(t) = x_0 \exp(i\,\omega t)$$

— but what does this ***mean***?

Complex Exponentials

Taylor Series Expansions for Exponential & Sinusoidal Functions:

Recall

$$\exp(z) = 1 + z + \tfrac{1}{2}z^2 + \tfrac{1}{3!}z^3 + \tfrac{1}{4!}z^4 + \cdots$$

$$\cos(z) = 1 - \tfrac{1}{2}z^2 + \tfrac{1}{4!}z^4 - \cdots$$

$$\sin(z) = z - \tfrac{1}{3!}z^3 + \cdots$$

What if $z = i\theta$?

$$\exp(i\theta) = 1 + i\theta - \tfrac{1}{2!}\theta^2 - \tfrac{1}{3!}i\theta^3 + \tfrac{1}{4!}\theta^4 + \cdots$$

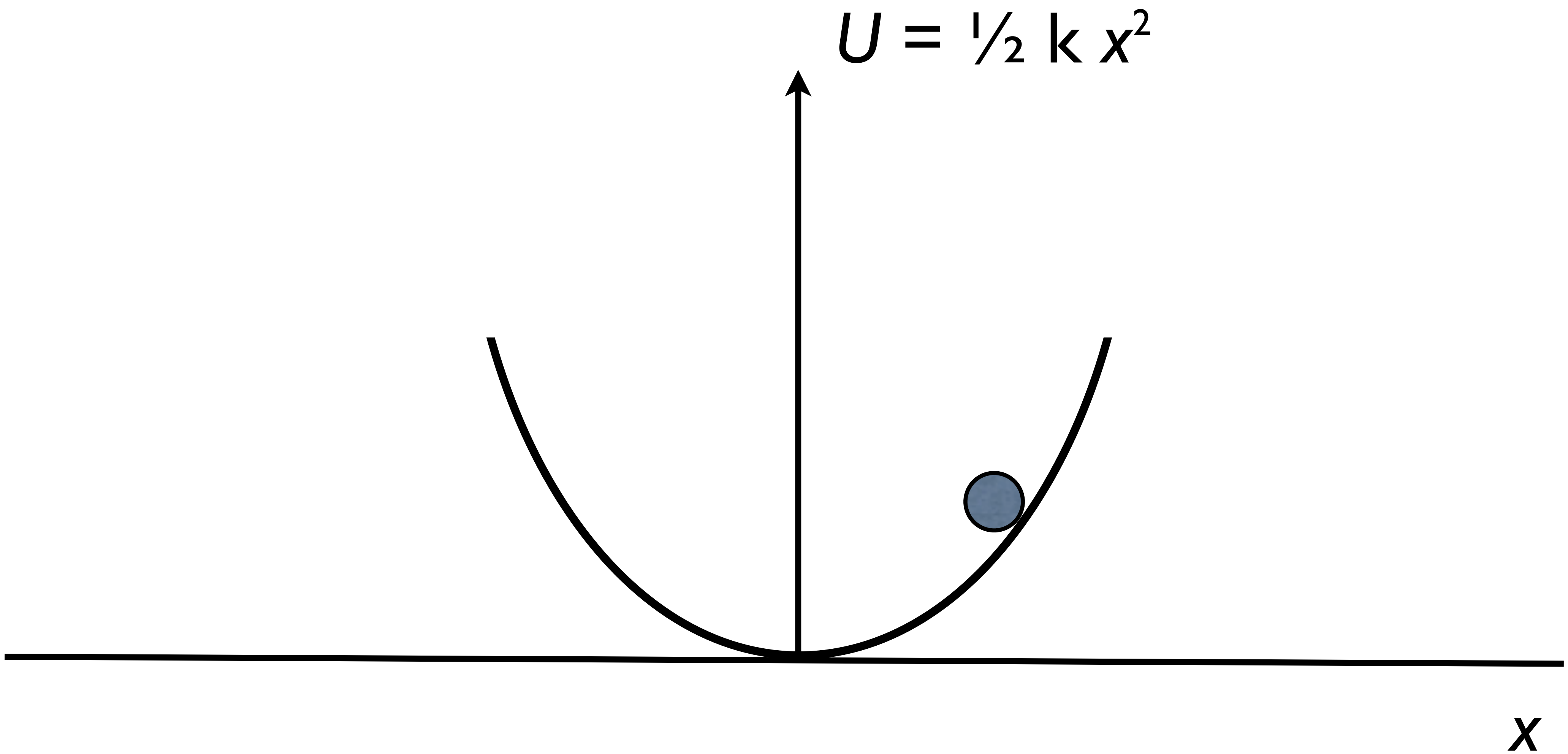
$$\cos(\theta) = 1 - \tfrac{1}{2!}\theta^2 + \tfrac{1}{4!}\theta^4 + \cdots$$

$$i\sin(\theta) = i\theta - \tfrac{1}{3!}i\theta^3 + \cdots$$

This means $e^{i\theta} = \cos\theta + i\sin\theta$

Don't tell anyone I told you about this!

Quadratic Potential Minimum



$$F = -dU/dx = -k\,x$$

Simple Harmonic Motion

Linear Restoring Force

(Hooke's Law)

$F = -k\,x$

⇕

Quadratic Potential Minimum

$U = \frac{1}{2}\,k\,x^2$

plus

Inertial Factor m

⇔

SHM

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

⇕

$$x = x_0 \cos(\omega t + \varphi)$$

$$v \equiv \frac{dx}{dt} = -\omega\, x_0 \sin(\omega t + \varphi)$$

$$a \equiv \frac{d^2x}{dt^2} = -\omega^2\, x_0 \cos(\omega t + \varphi)$$

$$\omega^2 = k/m$$

Damped Harmonic Motion:

Remember **viscous damping**?

$$d^2x/dt^2 = -\kappa\, dx/dt \quad \Leftrightarrow \quad v(t) = v_0 \exp(-\kappa t)$$

With a linear restoring force *and* viscous damping, the equation is

$$d^2x/dt^2 = -\kappa\, dx/dt - \omega^2 x$$

which still might be satisfied by $x(t) = x_0 \exp(Kt)$ with some K.

Let's try! Plug this $x(t)$ back into the equation, giving

$$K^2 x = -\kappa K x - \omega^2 x \qquad \text{or} \qquad K^2 + \kappa K + \omega^2 = 0$$

which has the solution $2K = -\kappa \pm \sqrt{\kappa^2 - 4\omega^2}$

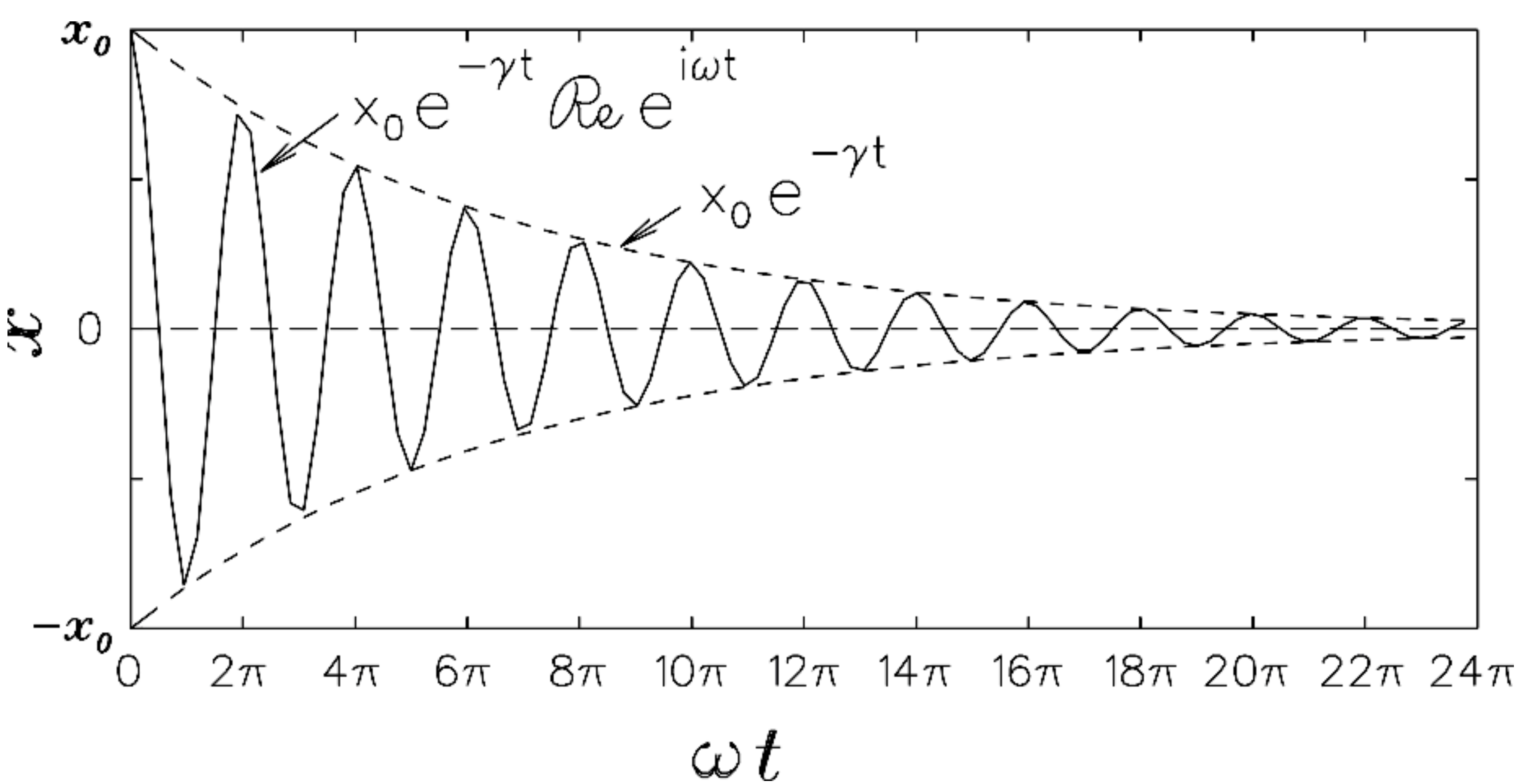
Again, *what does this **mean**?*

Damped Harmonic Motion:

If $2K = -\kappa \pm \sqrt{\kappa^2 - 4\omega^2}$ then

$$K = -\tfrac{1}{2}\kappa \pm i\,\omega\sqrt{1 - \tfrac{1}{4}\kappa^2/\omega^2}$$

$x(t) = x_0\, e^{-\frac{1}{2}\kappa} \exp(\pm i\,\omega' t)$ where $\omega' = \omega\sqrt{1 - \tfrac{1}{4}\kappa^2/\omega^2}$



The End

for now