The Emergence of Mechanics

What *use* are Newton's "Laws" of Mechanics? Even a glib answer to that question can easily fill a 1-year course, if you really want to know. My purpose here is merely to offer some hints of how people learned to apply Newton's Laws to different types of Mechanics problems, began to notice that they were repeating certain calculations over and over in certain wide classes of problems, and eventually thought of cute shortcuts that then came to have a life of their own. That is, in the sense of Michael Polanyi's *The Tacit Dimension*, a number of new paradigms *emerged* from the technology of *practical application* of Newton's Mechanics.

The mathematical process of emergence generally works like this: we take the SECOND LAW and transform it using a formal *mathematical identity* operation such as "Do the same thing to both sides of an equation and you get a new equation that is equally valid." Then we think up names for the quantities on both sides of the new equation and *presto!* we have a new paradigm. I will show three important example of this process, not necessarily the way they first were "discovered," but in such a way as to illustrate how such things can be done. But first we will need a few new mathematical tools.

Some Math Tricks

Differentials

We have learned that the symbols df and dx represent the *coupled changes* in $f(x)$ and x, in the limit where the change in x (and consequently also the change in f) become infinitesimally small. We call these symbols the **differentials** of f and x and distinguish them from Δf and Δx only in this sense: Δf and Δx can be any size, but df and dx are always *infinitesimal* — *i.e.* small enough so that we can treat $f(x)$ as a straight line over an interval only dx wide.

This does not change the interpretation of the representation $\frac{df}{dx}$ for the *derivative* of $f(x)$ with respect to x, but it allows us to think of these *differentials* df and dx as "normal" algebraic symbols that can be manipulated in the usual fashion. For instance, we can write

$$df = \left(\frac{df}{dx}\right) dx$$

which looks rather trivial in this form. However, suppose we give the derivative its own name:

$$g(x) \equiv \frac{df}{dx}$$

Then the previous equation reads

$$df = g(x)\, dx \qquad \text{or just} \qquad df = g\, dx$$

which can now be read as an expression of the *relationship between the two differentials* df and dx. Hold that thought.

As an example, consider our familiar kinematical quantities

$$a \equiv \frac{dv}{dt} \qquad \text{and} \qquad v \equiv \frac{dx}{dt}.$$

If we treat the differentials as simple algebraic symbols, we can invert the latter definition and write

$$\frac{1}{v} = \frac{dt}{dx}.$$

(Don't worry too much about what this "means" for now.) Then we can multiply the left side of the definition of a by $1/v$ and multiply the right side by dt/dx and get an equally valid equation:

$$\frac{a}{v} = \frac{dv}{dt} \cdot \frac{dt}{dx} = \frac{dv}{dx}$$

or, multiplying both sides by $v\,dx$,

$$a\,dx \;=\; v\,dv \tag{1}$$

which is a good example of a *mathematical identity*, in this case involving the *differentials* of distance and velocity. Hold *that* thought.

Antiderivatives

Suppose we have a function $g(x)$ which we know is the derivative [with respect to x] of some other function $f(x)$, but *we don't know which* — *i.e.* we know $g(x)$ explicitly but we don't know [yet] what $f(x)$ it is the derivative *of.* We may then ask the question, "What is the function $f(x)$ whose derivative [with respect to x] is $g(x)$?" Another way of putting this would be to ask, "What is the *antiderivative* of $g(x)$?"[1] Another name for the *antiderivative* is the *integral*, which is in fact the "official" version, but I like the former better because the name suggests how we go about "solving" one.[2]

For a handy example consider $g(x) = k\,x$. Then the *antiderivative* [integral] of $g(x)$ with respect to x is $f(x) = \frac{1}{2}\,k\,x^2 \;+\; f_0$ [where f_0 is some *constant*] because the derivative [with respect to x] of x^2 is $2x$ and the derivative of any constant is zero. Since any *combination* of constants is also a constant, it is equally valid to make the arbitrary constant term of the same *form* as the part which actually varies with x, *viz.* $f(x) = \frac{1}{2}\,k\,x^2 \;+\; \frac{1}{2}\,k\,x_0^2$. Thus f_0 is the same thing as $\frac{1}{2}\,k\,x_0^2$ and it is a matter of taste which you want to use.

Naturally we have a shorthand way of writing this. The *differential* equation

$$df = g(x)\;dx$$

can be turned into the *integral* equation

$$f(x) = \int_{x_0}^{x} g(x)\;dx \tag{2}$$

which reads, "$f(x)$ is the integral of $g(x)$ with respect to x from x_0 to x." We have used the rule that the *integral of the differential* of f [or any other quantity] is just the quantity itself,[3] in this case f:

$$\int df = f \tag{3}$$

Our example then reads

$$\int_{x_0}^{x} k\,x\,dx = k\int_{x_0}^{x} x\,dx = \frac{1}{2}\,k\,x^2 \;-\; \frac{1}{2}\,k\,x_0^2$$

where we have used the feature that any constant (like k) can be brought "outside the integral" — *i.e.* to the left of the integral sign $\int$.

Now let's use these new tools to transform Newton's SECOND LAW into something more comfortable.

[1] This is a lot like knowing that 6 is some number n multiplied by 2 and asking what n is. We figure this out by asking ourselves the question, "What do I have to multiply by 2 to get 6?" Later on we learn to call this "division" and express the question in the form, "What is $n = 6/2$?" but we might just as well call it "anti-multiplication" because that is how we solve it (unless it is too hard to do in our heads and we have to resort to some complicated technology like long division).

[2] Any introductory Calculus text will explain what an integral "means" in terms of visual pictures that the right hemisphere can handle easily: whereas the *derivative* of $f(x)$ is the *slope* of the curve, the *integral* of $g(x)$ is *the area under the curve*. This helps to visualize the integral as the limiting case of a *summation*: imagine the area under the curve of $g(x)$ from x_0 to x being divided up into N rectangular *columns* of equal width $\Delta x = \frac{1}{N}(x - x_0)$ and height $g(x_n)$, where $x_n = n\,\Delta x$ is the position of the $n^{\rm th}$ column. If N is a small number, then $\sum_{n=1}^{N} g(x_n)\,\Delta x$ is a crude approximation to the area under the smooth curve; but as N gets bigger, the columns get skinnier and the approximation becomes more and more accurate and is eventually (as $N \to \infty$) exact! This is the meaning of the integral sign:

$$\int_{x_0}^{x} g(x)\,dx \equiv \lim_{N\to\infty} \sum_{n=1}^{N} g(x_n)\,\Delta x$$

$$\text{where} \qquad \Delta x \equiv \frac{1}{N}(x - x_0) \qquad \text{and} \qquad x_n = n\,\Delta x.$$

Why do I put this nice graphical description in a footnote? Because we can understand most of the Physics applications of integrals by thinking of them as "antiderivatives" and because when we go to *solve* an integral we almost always do it by asking the question, "What function is this the derivative of?" which means *thinking* of integrals as antiderivatives. This is not a complete description of the mathematics, but it is sufficient for the purposes of this course. [See? We really *do* "deemphasize mathematics!"]

[3] This also holds for the integrals of differentials of *vectors*.

Impulse and Momentum

Multiplying a *scalar* times a vector is easy, it just changes its dimensions and length — *i.e.* it is transformed into a new *kind* of vector with new units but which is still in the same direction. For instance, when we multiply the vector *velocity* $\vec{\boldsymbol{v}}$ by the scalar *mass* m we get the vector *momentum* $\vec{\boldsymbol{p}} \equiv m\,\vec{\boldsymbol{v}}$. Let's play a little game with differentials and the SECOND LAW:

$$\vec{\boldsymbol{F}} = \frac{d\vec{\boldsymbol{p}}}{dt}.$$

Multiplying both sides by dt and integrating gives

$$\vec{\boldsymbol{F}}\,dt = d\vec{\boldsymbol{p}} \quad \Rightarrow \quad \int_{t_0}^{t} \vec{\boldsymbol{F}}\,dt = \int_{\vec{\boldsymbol{p}}_0}^{\vec{\boldsymbol{p}}} d\vec{\boldsymbol{p}} = \vec{\boldsymbol{p}} - \vec{\boldsymbol{p}}_0. \tag{4}$$

The left hand side of the final equation is the time integral of the net externally applied force $\vec{\boldsymbol{F}}$. This quantity is encountered so often in Mechanics problems [especially when $\vec{\boldsymbol{F}}$ is known to be an explicit function of time, $\vec{\boldsymbol{F}}(t)$] that we give it a *name*:

$$\int_{t_0}^{t} \vec{\boldsymbol{F}}(t)\,dt \;\equiv\; \text{IMPULSE due to applied force } \vec{\boldsymbol{F}} \tag{5}$$

Our equation can then be read as a sentence:

> "The *impulse* created by the net external force applied to a system is equal to the *momentum change* of the system."

Conservation of Momentum

The IMPULSE AND MOMENTUM law is certainly a rather simple transformation of Newton's SECOND LAW; in fact one may be tempted to think of it as a trivial restatement of the same thing. However, it is much simpler to *use* in many circumstances. The most useful application, surprisingly enough, is when there is *no* external force applied to the system and therefore *no* impulse and *no* change in momentum! In such cases the total momentum of the system does not change. We call this the LAW OF CONSERVATION OF MOMENTUM and use it much the same as Descartes and Huygens did in the days before Newton.[4]

Momentum conservation goes beyond Newton's FIRST LAW, though it may appear to be the same idea. Suppose our "system" [trick word, that!] consists not of *one* object but of *several*. Then the "net" [another one!] momentum of the system is the vector *sum* of the momenta of its components. This is where the power of momentum conservation becomes apparent. As long as there are no *external* forces, there can be as many forces as we like *between the component parts* of the system without having the slightest effect on their combined momentum. Thus, to take a macabre but traditional example, if we lob a hand grenade through the air, just after it explodes (before any of the fragments hit anything) all its pieces *taken together* still have the same net momentum as before the explosion.

The LAW OF CONSERVATION OF MOMENTUM is particularly important in analyzing the collisions of elementary particles. Since such collisions are the only means we have for performing experiments on the forces between such particles, you can bet that every particle physicist is very happy to have such a powerful (and simple-to-use!) tool.

Example: Volkwagen-Cadillac Scattering

Let's do a simple example in one dimension [thus avoiding the complications of adding and subtracting vectors] based on an apocryphal but possibly true story: A Texas Cadillac dealer once ran a TV ad showing a Cadillac running head-on into a parked Volkswagen Bug

[4] It should be remembered that René Descartes and Christian Huygens formulated the LAW OF CONSERVATION OF MOMENTUM *before* Newton's work on Mechanics. They probably deserve to be remembered as the First Modern Conservationists!

at 100 km/h. Needless to say, the Bug was squashed flat. Figs. 11.1 and 11.2 show a simplified sketch of this event, using the “before-and-after” technique with which our new paradigm works best. Figure 11.1 shows an *elastic* collision, in which the cars *bounce* off each other; Figure 11.2 shows a *plastic* collision in which they stick together. For quantitative simplicity

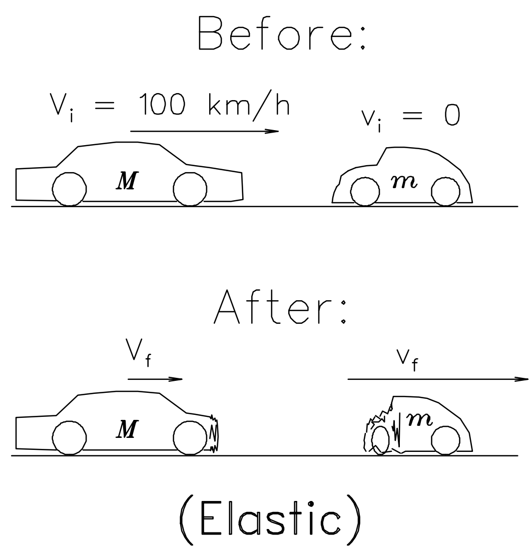

Figure 11.1 Sketch of a perfectly *elastic* collision between a Cadillac initially moving at 100 km/h and a parked Volkswagen Bug. For an elastic collision, the magnitude of the *relative* velocity between the two cars is the same before and after the collision. [The fact that the cars look “crunched” in the sketch reflects the fact that no actual collision between cars could ever be perfectly elastic; however, we will use this limiting case for purposes of illustration.]

we assume that the Cadillac has exactly twice the mass of the Bug ($M = 2m$). In both cases the net initial momentum of the “Caddy-Bug system” is $MV_i = 200m$, where I have omitted the “km/h” units of V_i, the initial velocity of the Caddy. Therefore, since all the forces act *between* the components of the system, the total momentum of the system is conserved and the net momentum *after* the collision must also be

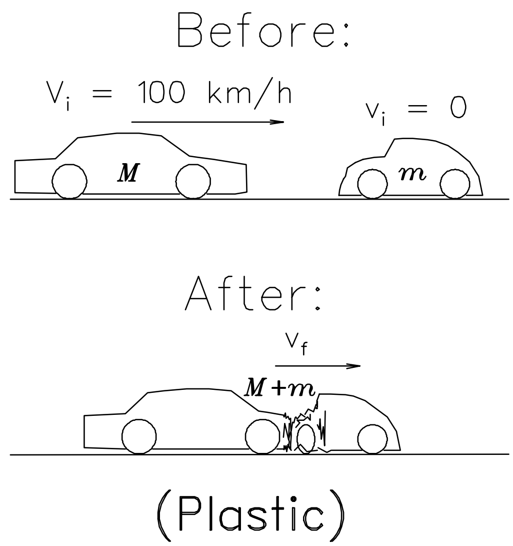

Figure 11.2 A perfectly inelastic or *plastic* collision in which the cars stick together and move as a unit after the collision.

$200m$.

In the *elastic* collision, the final *relative* velocity of the two cars must be the same as before the collision [this is one way of defining such a collision]. Thus if we assume (as on the drawing) that both cars move to the right after the collision, with velocities V_f for the Caddy and v_f for the Bug, then

$$v_f - V_f = 100 \qquad \text{or} \qquad v_f = V_f + 100.$$

Meanwhile the total momentum must be the same as initially:

$$MV_f + mv_f = 200m \qquad \text{or}$$

$$2mV_f + m(V_f + 100) = 200m$$

$$\text{or} \qquad 3mV_f = 100m$$

giving the final velocities

$$V_f = 33\frac{1}{3}\ \text{km/h} \qquad \text{and} \qquad v_f = 133\frac{1}{3}\ \text{km/h}.$$

In the *plastic* collision, the final system consists of both cars stuck together and moving to the right at a common velocity v_f. Again the total momentum must be the same as initially:

$$(M+m)v_f = 200m \qquad \text{or}$$

$$3mv_f = 200m \qquad \text{or}$$

$$v_f = 66\frac{2}{3} \text{ km/h.}$$

Several features are worth noting: first, the final velocity of the Bug after the *elastic* collision is actually *faster* than the Caddy was going when it hit! If the Bug then runs into a brick wall, well.... For anyone unfortunate enough to be inside one of the vehicles the severity of the consequences would be worst for the largest sudden change in the velocity of that vehicle — *i.e.* for the largest *instantaneous acceleration* of the passenger. This quantity is far larger *for both cars* in the case of the *elastic* collision. This is why "collapsibility" is an important safety feature in modern automotive design. You want your car to be completely demolished in a severe collision, with only the passenger compartment left intact, in order to minimize the recoil velocity. This may be annoyingly expensive, but it is nice to be around to enjoy the luxury of being annoyed!

Back to our story: The Cadillac dealer was, of course, trying to convince prospective VW buyers that they would be a lot safer in a Cadillac — which is undeniable, except insofar as the Bug's greater maneuverability and smaller "cross-section" [the size of the "target" it presents to other vehicles] helps to *avoid* accidents. However, the local VW dealer took exception to the Cadillac dealer's stated editorial opinion that Bugs should not be allowed on the road. To illustrate *his* point, he ran a TV ad showing a Mack truck running into a parked Cadillac at 100 km/h. The Cadillac was quite satisfactorily squashed and the VW dealer suggested sarcastically that perhaps everyone should be required by law to drive Mack trucks to enhance road safety. His point was well taken.

Centre of Mass Velocity

If we calculate the total momentum of a composite system and then divide by the total mass, we obtain the velocity of the system-as-a-whole, which we call the *velocity of the centre of mass.* If we imagine "running alongside" the system at this velocity we will be "in a reference frame moving with the centre of mass," where everything moves together and bounces apart [or whatever] with a very satisfying symmetry. Regardless of the internal forces of collisions, *etc.*, the centre of mass [CM] will be motionless in this reference frame. This has many convenient features, especially for calculations, and has the advantage that the inifinite number of other possible reference frames can all agree upon a common description in terms of the CM. Where exactly *is* the CM of a system? Well, wait a bit until we have defined *torques* and *rigid bodies*, and then it will be easy to show how to find the CM.

Work and Energy

We have seen how much fun it is to multiply the SECOND LAW by a scalar (dt) and integrate the result. What if we try multiplying through by a *vector*? As we have seen in the chapter on VECTORS, there are two ways to do this: the scalar or "dot" product $\vec{\boldsymbol{A}} \cdot \vec{\mathbf{B}}$, so named for the symbol $\cdot$ between the two vectors, which yields a *scalar* result, and the vector or "cross" product $\vec{\boldsymbol{A}} \times \vec{\mathbf{B}}$, whose name also reflects the appearance of the symbol $\times$ between the two vectors, which yields a *vector* result. The former is easier, so let's try it first.

In anticipation of situations where the applied force $\vec{\boldsymbol{F}}$ is an explicit function of the *position*[5]

[5] In the section on CIRCULAR MOTION we chose $\vec{\boldsymbol{r}}$ to denote the vector position of a particle in a circular orbit, using the centre of the circle as the origin for the $\vec{\boldsymbol{r}}$ vector. Here we are switching to $\vec{\boldsymbol{x}}$ to emphasize that the current description works equally well for any type of motion, circular or

$\vec{\boldsymbol{x}}$ — *i.e.* $\vec{\boldsymbol{F}}(\vec{\boldsymbol{x}})$ — let's try using a differential change in $\vec{\boldsymbol{x}}$ as our multiplier:

$$\begin{aligned}
\vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{x}} &= m\vec{\boldsymbol{a}} \cdot d\vec{\boldsymbol{x}} \\
&= m\frac{d\vec{\boldsymbol{v}}}{dt} \cdot d\vec{\boldsymbol{x}} \\
&= md\vec{\boldsymbol{v}} \cdot \frac{d\vec{\boldsymbol{x}}}{dt} \\
&= md\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{v}} \\
&= m\vec{\boldsymbol{v}} \cdot d\vec{\boldsymbol{v}}
\end{aligned}$$

where we have used the definitions of $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{v}}$ with a little shifting about of the differential dt and a reordering of the dot product [which we may always do] to get the right-hand side [RHS] of the equation in the desired form. A delightful consequence of this form is that it allows us to convert the RHS into an explicitly *scalar* form: $\vec{\boldsymbol{v}} \cdot d\vec{\boldsymbol{v}}$ is zero if $d\vec{\boldsymbol{v}} \perp \vec{\boldsymbol{v}}$ — *i.e.* if the change in velocity is *perpendicular* to the velocity itself, so that the *magnitude* of the velocity does not change, only the *direction*. [Recall the case of circular motion!] If, on the other hand, $d\vec{\boldsymbol{v}} \parallel \vec{\boldsymbol{v}}$, then the whole effect of $d\vec{\boldsymbol{v}}$ is to change the *magnitude* of $\vec{\boldsymbol{v}}$, not its direction. Thus $\vec{\boldsymbol{v}} \cdot d\vec{\boldsymbol{v}}$ is precisely a measure of the *speed* v times the differential *change* in speed, dv:

$$\vec{\boldsymbol{v}} \cdot d\vec{\boldsymbol{v}} = v\, dv \tag{6}$$

so that our equation can now be written

$$\vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{x}} = m\, v\, dv$$

and therefore

$$\int_{\vec{\boldsymbol{x}}_0}^{\vec{\boldsymbol{x}}} \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{x}} = m \int_{v_0}^{v} v\, dv = m\left(\frac{1}{2}v^2 - \frac{1}{2}v_0^2\right) \tag{7}$$

(Recall the earlier discussion of an equivalent *antiderivative*.)

otherwise. The two notations are interchangeable, but we tend to prefer $\vec{\boldsymbol{x}}$ when we are talking mainly about *rectilinear* (straight-line) motion and $\vec{\boldsymbol{r}}$ when we are referring our coordinates to some *centre* or *axis*.

Just to establish the connection to the mathematical identity $a\, dx = v\, dv$, we multiply that equation through by m and get $ma\, dx = mv\, dv$. Now, in *one dimension* (no vectors needed) we know to set $ma = F$ which gives us $F\, dx = mv\, dv$ or, integrating both sides,

$$\int_{x_0}^{x} F\, dx = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2$$

which is the same equation in one dimension.

OK, so what? Well, again this formula kept showing up over and over when people set out to solve certain types of Mechanics problems, and again they finally decided to recast the LAW in this form, giving new names to the left and right sides of the equation. We call $\vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{x}}$ the **work** dW done by exerting a force $\vec{\boldsymbol{F}}$ through a distance $d\vec{\boldsymbol{x}}$ [work is something we *do*] and we call $\frac{1}{2}mv^2$ the **kinetic energy** T. [kinetic energy is an *attribute* of a moving mass] Let's emphasize these definitions:

$$\int_{\vec{\boldsymbol{x}}_0}^{\vec{\boldsymbol{x}}} \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{x}} \equiv \Delta W\,, \tag{8}$$

the WORK done by $\vec{\boldsymbol{F}}(\vec{\boldsymbol{x}})$ over a path from $\vec{\boldsymbol{x}}_0$ to $\vec{\boldsymbol{x}}$, and

$$\frac{1}{2}mv^2 \equiv T\,, \tag{9}$$

the KINETIC ENERGY of mass m at speed v.

Our equation can then be read as a sentence:

> "When a force acts on a body, the *kinetic energy* of the body *changes* by an amount equal to the *work* done by the force exerted through a distance."

One nice thing about this "paradigm transformation" is that we have replaced a *vector* equation $\vec{\boldsymbol{F}} = m\,\vec{\boldsymbol{a}}$ by a *scalar* equation $\Delta W = \Delta T$. There are many situations in which the work done is easily calculated and the direction of the final velocity is obvious; one can then obtain the complete "final state" from the "initial state" in one quick step *without having to go through the details of what happens in between*. Another class of "before & after" problems solved!

Example: The Hill

Probably the most classic example of how the WORK AND ENERGY law can be used is the case of *a ball rolling down a frictionless hill*, pictured schematically in Fig. 11.3. Now, Galileo

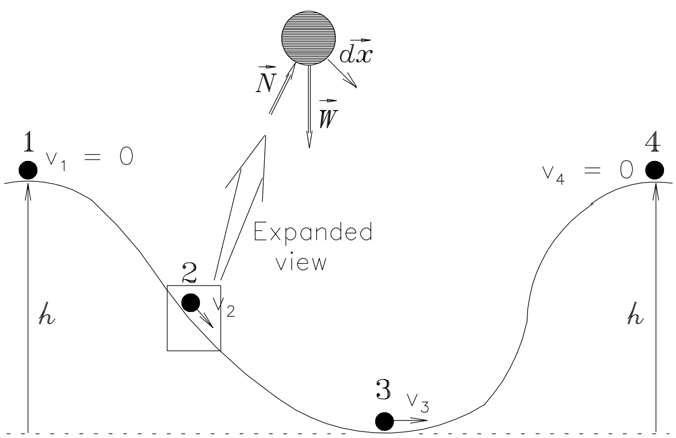

Figure 11.3 Sketch of a ball rolling down a frictionless hill. In position **1**, the ball is at rest. It is then given an infinitesimal nudge and starts to roll down the hill, passing position **2** on the way. At the bottom of the hill [position **3**] it has its maximum speed v_3, which is then dissipated in rolling up the other side of the hill to position **4**. Assuming that it stops on a slight slope at both ends, the ball will keep rolling back and forth forever.

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was fond of this example and could have given us a calculation of the final speed of the ball *for the case of a straight-line path* (*i.e.* the inclined plane); but he would have thrown up his hands at the picture shown in Fig. 11.3! Consider one spot on the downward slope, say position **2**: the FBD of the ball is drawn in the expanded view, showing the two forces $\vec{\boldsymbol{N}}$ and $\vec{\boldsymbol{W}}$ acting on the mass m of the ball.[6] Now, the ball does not jump off the surface or burrow into it, so the motion is strictly tangential to the hill at every point.[7] Meanwhile, a *frictionless* surface cannot, by definition, exert any force *parallel* to the surface; this is why the *normal force* $\vec{\boldsymbol{N}}$ is called a "normal" force — it is always normal [perpendicular] to the surface. So $\vec{\boldsymbol{N}} \perp d\vec{\boldsymbol{x}}$ which means that $\vec{\boldsymbol{N}} \cdot d\vec{\boldsymbol{x}} = 0$ and *the normal force does no work*! This is an important general rule. Only the gravitational force $\vec{\boldsymbol{W}}$ does any work on the mass m, and since $\vec{\boldsymbol{W}} = -m\,g\,\hat{y}$ is a constant downward vector [where we define the unit vector $\hat{y}$ as "up"], *it is only the downward component* of $d\vec{\boldsymbol{x}}$ that produces any work at all. That is, $\vec{\boldsymbol{W}} \cdot d\vec{\boldsymbol{x}} = -m\,g\,dy$, where dy is the component of $d\vec{\boldsymbol{x}}$ directed *upward*.[8] That is, no matter what angle the hill makes with the vertical at any position, at that position the work done by gravity in *raising* the ball a differential height dy is given by $dW = -m\,g\,dy$ [notice that gravity does *negative* work going *up*hill and *positive* work going *down*hill] and the net work done in raising the ball a total distance Δy is given by a rather easy integral:

$$\Delta W = -m\,g \int dy = -m\,g\,\Delta y$$

where Δy is the height that the ball is *raised* in the process. By our LAW, this must be equal to the change in the *kinetic energy* $T \equiv \frac{1}{2}mv^2$ so that

$$\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 \;=\; -\,m\,g\,\Delta y. \qquad (10)$$

This formula governs both uphill rolls, in which Δy is positive and the ball slows down, and downhill rolls in which Δy is negative and the ball speeds up. For the example shown in Fig. 11.3 we start at the top with $v_0 = v_1 = 0$ and roll down to position **3**, dropping the

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[6] It is unfortunate that the conventional symbol for the *weight*, $\vec{\boldsymbol{W}}$, uses the same letter as the conventional symbol for the *work*, W. I will try to keep this straight by referring to the weight always and only in its *vector* form and reserving the *scalar* W for the work. But this sort of difficulty is eventually inevitable.

[7] For now, I specifically *exclude* cases where the ball gets going so fast that it *does* get airborne at some places.

[8] Alas, another unfortunate juxtaposition of symbols! We are using $d\vec{\boldsymbol{x}}$ to describe the differential *vector* position change and dy to describe the *vertical component* of $d\vec{\boldsymbol{x}}$. Fortunately we have no cause to talk about the horizontal component in this context, or we might wish we had used $d\vec{\boldsymbol{r}}$ after all!

height by an amount h in the process, so that the maximum speed (at position **3**) is given by

$$\frac{1}{2}mv_3^2 = mgh \qquad \text{or} \qquad v_3 = \sqrt{2gh}.$$

On the way up the other side the process exactly reverses itself [though the *details* may be completely different!] in that the altitude once again increases and the velocity drops back to zero.

The most pleasant consequence of this paradigm is that as long as the surface is truly frictionless, *we never have to know any of the details about the descent* to calculate the velocity at the bottom! The ball can drop straight down, it can roll up and down any number of little hills [as long as none of them are higher than its original position] or it can even roll through a tunnel or "black box" whose interior is hidden and unknown — and as long as I guarantee a frictionless surface you can be confident that it will come out the other end at the same speed as if it had just fallen the same vertical distance straight down. The *direction* of motion at the bottom will of course always be tangential to the surface.

For me it seems impossible to imagine the ball rolling up and down the hill without starting to think in terms of kinetic energy being *stored up* somehow and then automatically re-emerging from that storage as fresh kinetic energy. But I have already been indoctrinated into this way of thinking, so it is hard to know if this is really a compelling metaphor or just an extremely successful one. You be the judge. I will force myself to hold off talking about *potential energy* until I have covered the second prototypical example of the interplay between work and energy.

The Stretched Spring

The *spring* embodies one of Physics' premiere paradigms, the *linear restoring force*. That is, a

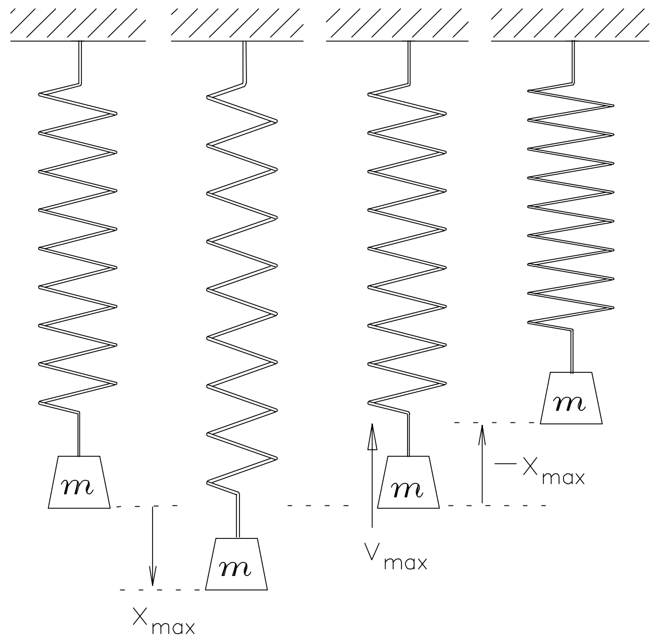

Figure 11.4 Sketch of a mass on a spring. In the leftmost frame the mass m is at rest and the spring is in its equilibrium position (*i.e.* neither stretched nor compressed). [If gravity is pulling the mass down, then in the equilibrium position the spring is stretched just enough to counteract the force of gravity. The equilibrium position can still be taken to define the $x = 0$ position.] In the second frame, the spring has been gradually pulled down a distance $x_{\rm max}$ and the mass is once again at rest. Then the mass is released and accelerates upward under the influence of the spring until it reaches the equilibrium position again [third frame]. This time, however, it is moving at its maximum velocity $v_{\rm max}$ as it crosses the centre position; as soon as it goes higher, it *compresses* the spring and begins to be *decelerated* by a linear restoring force in the opposite direction. Eventually, when $x = -x_{\rm max}$, all the kinetic energy has been been stored back up in the compression of the spring and the mass is once again instantaneously at rest [fourth frame]. It immediately starts moving downward again at maximum acceleration and heads back toward its starting point. In the absence of friction, this cycle will repeat forever.

force which disappears when the system in question is in its "equilibrium position" x_0 [which we will define as the $x = 0$ position ($x_0 \equiv 0$) to make the calculations easier] but increases as x moves away from equilibrium, in such a way that the *magnitude* of the force F is proportional to the displacement from equilibrium [F is *linear* in x] and the *direction* of F is such as to try to *restore* x to the original position. The constant of proportionality is called the *spring constant*, always written k. Thus (using vector notation to account for the directionality)

$$\vec{\boldsymbol{F}} = -k\,\vec{\boldsymbol{x}} \tag{11}$$

which is the mathematical expression of the concept of a *linear restoring force*. This is definitely one to remember.

Keeping in mind that the $\vec{\boldsymbol{F}}$ given above is the force exerted *by the spring* against anyone or anything trying to stretch or compress it. If you are that stretcher/compressor, the force *you* exert is $-\vec{\boldsymbol{F}}$. If you *do work on the spring*[9] by stretching or compressing it[10] by a differential displacement $d\vec{\boldsymbol{x}}$ from equilibrium, the differential amount of work done is given by

$$dW \;=\; -\vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{x}} \;=\; k\,\vec{\boldsymbol{x}} \cdot d\vec{\boldsymbol{x}} \;=\; k\,x\,dx$$

which we can integrate from $x = 0$ (the equilibrium position) to x (the final position) to get the net work W:

$$W = k \int_0^x x\,dx = \frac{1}{2} k\,x^2 \tag{12}$$

Once you let go, *the spring will do the same amount of work back* against the only thing trying to impede it — namely, the inertia of the mass m attached to it. This can be used with the Work and Energy Law to calculate the speed $v_{\max}$ in the third frame of Fig. 11.4: since $v_0 = 0$,

$$\frac{1}{2} m\,v_{\max}^2 = \frac{1}{2} k\,x_{\max}^2 \qquad \text{or} \qquad v_{\max}^2 = \frac{k}{m}\,x_{\max}^2$$

$$\text{or} \qquad v_{\max} = \sqrt{\frac{k}{m}}\;|x_{\max}|$$

where $|x_{\max}|$ denotes the *absolute value* of $x_{\max}$ (*i.e.* its magnitude, always positive). Note that this is a relationship between the *maximum* values of v and x, which occur *at different times* during the process.

Love as a Spring

Few other paradigms in Physics are so easy to translate into "normal life" terms as the *linear restoring force*. As a whimsical example, consider an intimate relationship between two lovers. In this case x can represent "emotional distance" — a difficult thing to quantify but an easy one to imagine. There is some *equilibrium distance* x_0 where at least one of the lovers is most comfortable[11] — this time, just to show how it works, we will not choose x_0 to be the zero position of x but leave it in the equations explicitly. When circumstances (usually *work*) *force* a greater emotional *distance* for a while, the lover experiences a sort of *tension* that *pulls* him or her back closer to the beloved. This is a perfect analogy to the linear restoring force:

$$F = -k\,(x - x_0)$$

What few people seem to recognize is that this "force," like any linear restoring force, is symmetric: it works the same in both directions, too far apart and too close. When circumstances permit a return to greater closeness, the lover rushes back to the beloved (figuratively — we are talking about emotional distance x here!)

[9] It is important to keep careful track of *who* is doing work on *whom*, especially in this case, because if you are careless the minus signs start jumping around and multiplying like cockroaches!

[10] It doesn't matter which — if you stretch it out you have to *pull* in the same direction as it moves, while if you compress it you have to *push* in the direction of motion, so either way the force and the displacement are in the same direction and you do *positive work* on the spring.

[11] Sadly, x_0 is not always the same for both partners in the relationship; this is a leading cause of *tension* in such cases. [Doesn't this metaphor extend gracefully?]

and very often "overshoots" the equilibrium position x_0 to get temporarily *closer* than is comfortable. The natural *repulsion* that then occurs is no cause for dismay — you can't really have an attraction without it — but some people seem surprised to discover that the *attraction* that binds them to their beloved does not just keep acting no matter how close they get; they are very upset that x cannot just keep getting closer and closer without limit.[12] In later chapters I will have much more to say about the *oscillatory* pattern that gets going [see Fig. 11.4] when the *overshoot* is allowed to occur without any *friction* to dissipate the energy stored in the stretched spring [a process known as *damping*]. But first I really must pick up another essential paradigm that has been begging to be introduced.

Potential Energy

Imagine yourself on skis, poised motionless at the top of a snow-covered hill: one way or another, you are deeply aware of the *potential* of the hill to increase your speed. In Physics we like to think of this obvious capacity as the *potential* for *gravity* to increase your *kinetic energy*. We can be quantitative about it by going back to the bottom of the hill and recalling the long trudge *uphill* that it took to get to the top: this took a lot of *work*, and we know the formula for how much: in raising your elevation by a height h you did an amount of work $W = mgh$ "against gravity" [where m is your mass, of course]. That work is now somehow "stored up" because if you slip over the edge it will all come back to you in the form of kinetic energy! What could be more natural than to think of that "stored up work" as *gravitational potential energy*

$$V_g = m\,g\,h \tag{13}$$

which will all turn into kinetic energy if we allow h to go back down to zero?[13]

We can then picture a skier in a bowl-shaped valley zipping down the slope to the bottom $[V_g \to T]$ and then coasting back up to stop at the original height $[T \to V_g]$ and (after a skillful flip-turn) heading back downhill again $[V_g \to T]$. *In the absence of friction*, this could go on forever: $V_g \to T \to V_g \to T \to V_g \to T \to \ldots$.

The case of the spring is even more compelling, in its way: if you push in the spring a distance x, you have done some work $W = \frac{1}{2}k\,x^2$ "against the spring." If you let go, this work "comes back at you" and will accelerate a mass until all the stored energy has turned into kinetic energy. Again, it is irresistible to call that "stored spring energy" the *potential energy* of the spring,

$$V_s = \frac{1}{2}k\,x^2 \tag{14}$$

and again the scenario after the spring is released can be described as a perpetual cycle of $V_s \to T \to V_s \to T \to V_s \to T \to \ldots$.

Conservative Forces

Physicists so love their ENERGY paradigm that it has been elevated to a higher status than the original SECOND LAW from which it was derived! In orer to make this switch, of course, we had to invent a way of making the *reverse derivation* — *i.e.* obtaining the vector *force* $\vec{\boldsymbol{F}}$ exerted "spontaneously" by the system in question from the scalar *potential energy* V of the system. Here's how: in one dimension we can forget the vector stuff and just juggle the differentials

[12] I suspect that such foolishness is merely an example of single-valued logic [closer = better] obsessively misapplied, rather than some more insidious psychopathology. But I could be wrong!

[13] The choice of a *zero point* for V_g is arbitrary, of course, just like our choice of where $h = 0$. This is not a problem if we allow *negative* potential energies [which we do!] since it is only the *change* in potential energy that appears in any actual mechanics problem.

in $dW_{\rm me} = F_{\rm me}\, dx$, where the $W_{\rm me}$ is the work I do in exerting a force $F_{\rm me}$ "against the system" through a distance dx. Assuming that *all* the work I do against the system is *conserved* by the system in the form of its *potential energy* V, then $dV = dW_{\rm me}$. On the other hand, the force F exerted *by* the system [*e.g.* the force exerted by the spring] is the equal and opposite reaction force to the force I exert: $F = -F_{\rm me}$. The law for *conservative forces in one dimension* is then

$$F = -\frac{dV}{dx} \tag{15}$$

That is, the force of (*e.g.*) the spring is *minus* the *rate of change of the potential energy with distance*.

In three dimensions this has a little more complicated form, since $V(\vec{\boldsymbol{x}})$ could in principle vary with all three components of $\vec{\boldsymbol{x}}$: x, y and z. We can talk about the three components independently,

$$F_x = -\frac{\partial V}{\partial x}, \quad F_y = -\frac{\partial V}{\partial y} \quad \text{and} \quad F_z = -\frac{\partial V}{\partial z}$$

where the notation ∂ is used to indicate derivatives with respect to *one* variable of a function of *several* variables [here $V(x,y,z)$] *with the other variables held fixed*. We call $\partial V/\partial x$ the *partial derivative* of V with respect to x. In the same spirit that moved us to invent vector notation in the first place [*i.e.* making the notation more compact], we use the *gradient operator*

$$\vec{\boldsymbol{\nabla}} \equiv \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} \tag{16}$$

to express the three equations above in one compact form:

$$\vec{\boldsymbol{F}} = -\vec{\boldsymbol{\nabla}} V \tag{17}$$

The *gradient* is easy to visualize in *two* dimensions: suppose you are standing on a *real* hill. Since your *height* $h \equiv z$ is actually proportional to your gravitational potential energy V_g, it is perfectly consistent to view the actual hill as a *graph* of the function $V_g(x,y)$ of East-West coordinate x and North-South coordinate y. In this picture, looking down on the hill from above, the direction of the gradient $\vec{\boldsymbol{\nabla}} V_g$ is *uphill*, and the magnitude of the gradient is the *slope* of the hill at the position where the gradient is evaluated. The nice feature is that $\vec{\boldsymbol{\nabla}} V_g$ will automatically point "straight up the hill" — *i.e.* in the steepest direction. Thus $-\vec{\boldsymbol{\nabla}} V_g$ points "straight downhill" — *i.e.* in the direction a marble will roll if it is released at that spot! There are lots of neat tricks we can play with the gradient operator, but for now I'll leave it to digest.

Friction

What about not-so-conservative forces? In the real world a lot of energy gets *dissipated* through what is loosely known as *friction*. Nowhere will you find an entirely satisfactory definition of precisely what friction is, so I won't feel guilty about using the cop-out and saying that it is the cause of all work that does *not* "get stored up as potential energy." That is, when I do work against frictional forces, it will not reappear as kinetic energy when I "let go."

Where does it go? We have already started getting used to the notion that energy is *conserved*, so it is disturbing to find some work just being *lost*. Well, relax. The energy dissipated by work against friction is still around in the form of *heat*, which is something like *disordered potential and kinetic energy*.[14] We will talk more about heat a few chapters later.

Torque and Angular Momentum

Finally we come to the formally trickiest transformation of the SECOND LAW, the one involving the *vector product* (or "cross product") of $\vec{\boldsymbol{F}}$ with the distance $\vec{\boldsymbol{r}}$ away from some *ori-*

[14] [Not quite, but you can visualize lots of little atoms wiggling and jiggling seemingly at random — that's heat, sort of.]

gin[15] "O." Here goes:

$$\vec{\boldsymbol{r}} \times \left[d\frac{\vec{\boldsymbol{p}}}{dt} = \vec{\boldsymbol{F}} \right] \qquad \text{gives} \qquad \vec{\boldsymbol{r}} \times \frac{d\vec{\boldsymbol{p}}}{dt} = \vec{\boldsymbol{r}} \times \vec{\boldsymbol{F}}$$

Now, the distributive law for derivatives applies to cross products, so

$$\frac{d}{dt}\left[\vec{\boldsymbol{r}} \times \vec{\boldsymbol{p}}\right] = \frac{d\vec{\boldsymbol{r}}}{dt} \times \vec{\boldsymbol{p}} + \vec{\boldsymbol{r}} \times \frac{d\vec{\boldsymbol{p}}}{dt}$$

but

$$\frac{d\vec{\boldsymbol{r}}}{dt} \equiv \vec{\boldsymbol{v}} \qquad \text{and} \qquad \vec{\boldsymbol{p}} \equiv m\,\vec{\boldsymbol{v}}$$

$$\text{so} \qquad \frac{d\vec{\boldsymbol{r}}}{dt} \times \vec{\boldsymbol{p}} = m\left(\vec{\boldsymbol{v}} \times \vec{\boldsymbol{v}}\right) = 0$$

because the cross product of any vector with *itself* is zero.[16] Therefore

$$\frac{d}{dt}\left[\vec{\boldsymbol{r}} \times \vec{\boldsymbol{p}}\right] = \vec{\boldsymbol{r}} \times \vec{\boldsymbol{F}}.$$

If we define two new entities,

$$\vec{\boldsymbol{r}} \times \vec{\boldsymbol{p}} \equiv \vec{\boldsymbol{L}}_O, \tag{18}$$

the *Angular Momentum* about O

and

$$\vec{\boldsymbol{r}} \times \vec{\boldsymbol{F}} \equiv \vec{\boldsymbol{\Gamma}}_O, \tag{19}$$

the *Torque* generated by $\vec{\boldsymbol{F}}$ about O,

then we can write the above result in the form

$$\frac{d\vec{\boldsymbol{L}}_O}{dt} = \vec{\boldsymbol{\Gamma}}_O \tag{20}$$

This equation looks remarkably similar to the SECOND LAW. In fact, it is the *rotational analogue* of the SECOND LAW. It says that

> "The rate of change of the *angular momentum* of a body *about the origin* O is equal to the *torque* generated by forces acting about O."

[15] Note that everything we discuss in this case will be *with reference to the chosen origin* O, which may be chosen arbitrarily but must then be carefully remembered!

[16] Remember from the chapter on VECTORS that only the *perpendicular* parts of two vectors contribute to the cross product. *Any* two *parallel* vectors have *zero cross product*. A vector crossed with itself is the simplest example.

So what? Well, *if we choose the origin cleverly* this "new" Law gives us some very nice generalizations. Consider for instance an example which occurs *very* often in physics: the *central force*.

Central Forces

Many [maybe even most] forces in nature are directed toward [or away from] some "source" of the force. An obvious example is Newton's Universal Law of Gravitation, but there are many others evident, especially in elementary particle physics.[17] We call these forces "central" because if we regard the point toward [or away from] which the force points as the *centre* (or *origin* O) of our coordinate system, from which the position vector $\vec{\boldsymbol{r}}$ is drawn, the cross product between $\vec{\boldsymbol{r}}$ and $\vec{\boldsymbol{F}}$ (which is along $\hat{r}$) is always zero. That is,

> "A *central force* produces *no torque* about the centre; therefore the *angular momentum* about the centre *remains constant* under a central force."

This is the famous Law of CONSERVATION OF ANGULAR MOMENTUM. Note the limitation on its applicability.

The Figure Skater

Again, so what? Well, there are numerous examples of central forces in which angular momentum conservation is used to make sense of otherwise counterintuitive phenomena. For instance, consider the classic image of the *figure skater* doing a pirouette: she starts spinning with hands

[17] For instance, the *electrostatic* force between two point *charges* obeys exactly the same "inverse square law" as gravitation, except with a much stronger constant of proportionality and the inclusion of both positive and negative charges. We will have lots more to do with that later on!

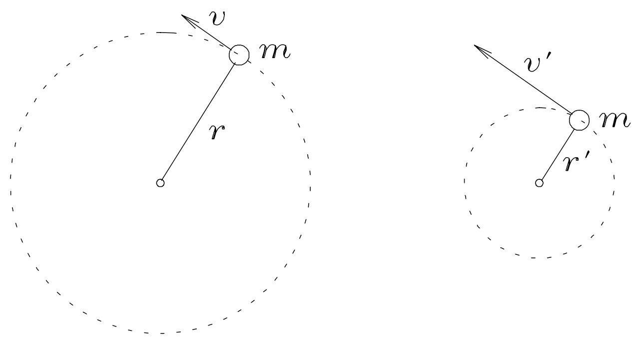

Figure 11.5 A contrived central-force problem. The ball swings around (without friction, of course) on the end of a string fixed at the origin O. The central force in the string cannot generate any torque about O, so the angular momentum $L_O = mvr$ about O must remain constant. As the string is pulled in slowly, the radius r gets shorter so the momentum $p = mv = mr\omega$ has to increase to compensate.

and feet as far extended as possible, then pulls them in as close to her body. As a result, even though no *torques* were applied, she spins much faster. Why? I can't draw a good figure skater, so I will resort to a cruder example [shown in Fig. 11.5] that has the same qualitative features: imagine a ball (mass m) on the end of a string that emerges through a hole in an axle which is held rigidly fixed. The ball is swinging around in a circle in the end of the string. For an initial radius r and an initial velocity $v = r\omega$, the initial momentum is $mr\omega$ and the angular momentum about O is $L_O = mvr = mr^2\omega$. Now suppose we pull in the string until $r' = \frac{1}{2}r$. To keep the same L_O the momentum (and therefore the velocity) must increase by a factor of 2, which means that the *angular* velocity $\omega' = 4\omega$ since the ball is now moving at twice the speed but has only half as far to go around the circumference of the circle. The *period* of the "orbit" has thus *decreased* by a factor of four!

Returning to our more æsthetic example of the figure skater, if she is able to pull in all her mass a factor of 2 closer to her centre (on average) then she will spin 4 times more rapidly in the sense of revolutions per second or "Hertz" (Hz).

Kepler Again

A more formal example of the importance of the Law of Conservation of Angular Momentum under Central Forces is in its application to Celestial Mechanics, where the gravitational attraction of the Sun is certainly a classic central force. If we always use the Sun as our origin O, neglecting the influence of other planets and moons, the orbits of the planets must obey Conservation of Angular Momentum about the Sun. Suppose we draw a radius r from the Sun to the planet in question, as in Fig. 11.6. The rate

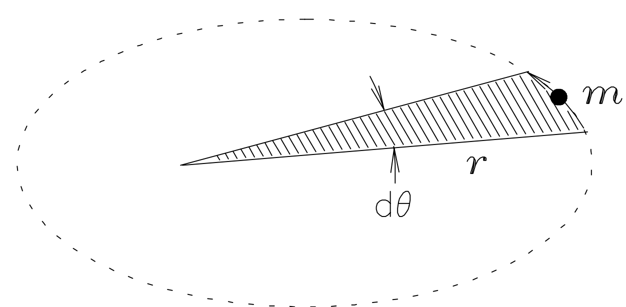

Figure 11.6 A diagram illustrating the *areal velocity* of an orbit. A planet (mass m) orbits the Sun at a distance r. the shaded area is equal to $\frac{1}{2}r \times r\, d\theta$ in the limit of infinitesimal intervals [*i.e.* as $d\theta \to 0$]. The *areal velocity* [*rate* at which this area is swept out] is thus $\frac{1}{2}r^2 d\theta/dt = \frac{1}{2}r^2\omega$.

at which this radius vector "sweeps out area" as the planet moves is $\frac{1}{2}r^2\omega$, whereas the *angular momentum about the Sun* is $mr^2\omega$. The two quantities differ only by the constants $\frac{1}{2}$ and m; therefore Kepler's *empirical* observation that the planetary orbits have constant "areal velocity" is *equivalent* to the requirement that the angular momentum about the Sun be a conserved quantity.

Rigid Bodies

Despite the fact that all Earthly matter is composed mostly of empty space sprinkled lightly with tiny bits of mass called atomic nuclei and even tinier bits called electrons, the forces between these bits are often so enormous that they hold the bits rigidly locked in a regular array called a *solid*. Within certain limits these arrays behave as if they were inseperable and perfectly rigid. It is therefore of some practical importance to develop a body of understanding of the behaviour of such *rigid bodies* under the influence of external forces. This is where the equations governing *rotation* come in.

A Moment of Inertia, Please!

Just as in the *translational* [straight-line motion] part of Mechanics there is an inertial factor m which determines how much p you get for a given $v \equiv \dot{x}$ and how much $a \equiv \dot{v} \equiv \ddot{x}$ you get for a given F, so in *rotational* Mechanics there is an *angular analogue* of the inertial factor that determines how much L_O you get for a given $\omega \equiv \dot{\theta}$ and how much $\alpha \equiv \dot{\omega}$ you get for a given Γ_O. This angular inertial factor is called the *moment of inertia* about O [we must always specify the origin about which we are defining torques and angular momentum] and is written I_O with the prescription

$$I_O = \int r_\perp^2 \, dm \tag{21}$$

where the integral represents a *summation* over all little "bits" of mass dm [we call these "mass elements"] which are distances $r_\perp$ away from an *axis* through the point O. Here we discover a slight complication: $r_\perp$ is measured from the *axis*, not from O itself. Thus a mass element dm that is a long way from O but right on the axis will contribute nothing to I_O. This continues to get more complicated until we have a complete description of Rotational Mechanics with I_O as a *tensor* of inertia and lots of other stuff I will never use again in this course. I believe I will stop here and leave the finer points of Rotational Mechanics for later Physics courses!

Rotational Analogies

It is, however, worth remembering that all the now-familiar [?] paradigms and equations of Mechanics come in "rotational analogues:"

Linear Version	**Angular Version**	**Name**
x	θ	angle
$\dot{x} \equiv v$	$\dot{\theta} \equiv \omega$	angular velocity
$\ddot{x} \equiv \dot{v} \equiv a$	$\ddot{\theta} \equiv \dot{\omega} \equiv \alpha$	angular acceleration
m	I_O	moment of inertia
$p = m\,v$	$L_O = I_O\omega$	angular momentum
F	Γ_O	torque
$\dot{p} = F$	$\dot{L}_O = \Gamma_O$	SECOND LAW
$T = \frac{1}{2}mv^2$	$T = \frac{1}{2}I_O\omega^2$	rotational kinetic energy
$dW = F\,dx$	$dW = \Gamma d\theta$	rotational work
$F = -kx$	$\Gamma = -\kappa\theta$	torsional spring law
$V_s = \frac{1}{2}k\,x^2$	$V_s = \frac{1}{2}\kappa\,\theta^2$	torsional potential energy

Statics

The enormous technology of *Mechanical Engineering* can be in some naïve sense be reduced to the two equations

$$\dot{\vec{\boldsymbol{p}}} = \vec{\boldsymbol{F}} \qquad \text{and} \qquad \dot{\vec{\boldsymbol{L}}}_O = \vec{\boldsymbol{\Gamma}}_O .$$

Whole courses are taught on what amounts to these two equations and the various tricks for solving them in different types of situations. Fortunately, this isn't one of them! Just to give a flavour, however, I will mention the basic problem-solving technique of *Statics*, the science of things that are sitting still![18] That means $\dot{\vec{\boldsymbol{p}}} = 0$ and $\dot{\vec{\boldsymbol{L}}}_O = 0$ so that the relevant equations are now

$$\sum \vec{\boldsymbol{F}} \;=\; 0 \qquad \text{and} \qquad \sum \vec{\boldsymbol{\Gamma}}_O \;=\; 0$$

where the $\sum$ [summation] symbols emphasize that there is never just *one* force or *one* torque acting on a rigid body in equilibrium; if there were, it (the force or torque) would be unbalanced and acceleration would inevitably result!

To solve complex three-dimensional Statics problems it is often useful to back away from our nice tidy vector formalism and explicitly write out the "equations of equilibrium" in terms of the components of the forces along the $\hat{x}, \hat{y}$ and $\hat{z}$ directions as well as the torques about the x, y and z axes [which meet at the origin O]:

$$\sum F_x \;=\; 0 \qquad \sum \Gamma_x \;=\; 0 \qquad (22)$$

$$\sum F_y \;=\; 0 \qquad \sum \Gamma_y \;=\; 0 \qquad (23)$$

$$\sum F_z \;=\; 0 \qquad \sum \Gamma_z \;=\; 0 \qquad (24)$$

If you have some civil engineering to do, you can work it out with these equations. Or hire an Engineer. I suggest the latter.

[18] This is pretty boring from a Physicist's point of view, but even Physicists are grateful when bridges do not collapse.

Physics as Poetry

This has been a long chapter; it needs some summary remarks. All I have set out to do here is to introduce the paradigms that emerged from Newton's SECOND LAW through mathematical identity transformations. This process of *emergence* seems almost miraculous sometimes because by a simple [?] rearrangement of previously defined concepts we are able to create new *meaning* that wasn't there before! This is one of the ways Physics bears a family resemblance to Poetry and the other Arts. The Poet also juxtaposes familiar images in a new way and creates meaning that no one has ever seen before; this is the finest product of the human mind and one of the greatest inspirations to the human spirit.

In Physics, of course, the process is more sluggish, because we insist on working out all the ramifications of every new paradigm shift and evaluating its elegance and utility in some detail before we decide to "go with it." This explains why it is so easy to describe just how the concepts introduced in this chapter *emerged* from Newton's Mechanics, but not so easy to tidily describe the consequences (or even the *nature*) of more recent paradigm shifts whose implications are still being discovered. There is a lot of technical overhead to creativity in Physics.

A Physics paradigm shift is a profound alteration of the way Physicists see the world; but what do the rest of us care? It can be argued that such shifts have effects on our Reality even if we choose to exclude Physics from our immediate awareness. Examples of this are plentiful even in Classical Mechanics, but the first dramatic social revolution that can be clearly seen to have arisen largely from the practical consequences of breakthroughs in Physics was the Industrial Revolution, the origins of which will be discussed in the chapter on Thermal Physics.